ALGEBRAIC GEOMETRY - MIDSEMESTER EXAMINATION

Attempt all questions - 10 a.m. to 1 p.m., 2nd March 2012 - Total Marks:50

In all questions below, the field k is algebraically closed, unless otherwise mentioned.

(1) Let $X, Y \subset k^n$ be affine algebraic sets.
  (a) Prove that $\sqrt{I(X) + I(Y)} = I(X \cap Y)$.
  (b) Give an example to show that $I(X) + I(Y)$ can be strictly contained in $I(X \cap Y)$.
  (c) If k is **not** algebraically closed, then give an example to show that $\sqrt{I(X) + I(Y)}$ can be strictly contained in $I(X \cap Y)$.
  (4+3+3=10 marks)

(2) Consider the map $\phi : k^2 \to k^2$ given by $(x, y) \mapsto (x, xy)$, and let $Im(\phi)$ denote the image of ϕ.
  (a) Describe $Im(\phi)$ as a set.
  (b) Is $Im(\phi)$ a closed subset of k^2 in the Zariski topology?
  (c) Is $Im(\phi)$ an open subset of k^2 in the Zariski topology?
  (d) What is the induced map on coordinate rings, $\phi^* : k[x, y] \to k[x, y]$ (here $\Gamma(k^2) \cong k[x, y]$)?
  (2+3+3+3=11 marks)

(3) Let $I \subset k[x_1, \dots, x_n]$ be an ideal.
  (a) Prove that $V(I) \subset k^n$ is a finite set, if and only if, $k[x_1, \dots, x_n]/I$ is a finite dimensional vector space over k.
  (b) Show that if (a) holds, $card(V(I)) \leq dim_k(k[x_1, \dots, x_n]/I)$, and give an example where strict inequality holds.
  (c) If k is **not** algebraically closed, then is (a) still true? Justify your answer.
  (6+3+3=12 marks)

(4) Recall that a *line* in $\mathbb{P}^2$ is a subset of the form $V(F) \subset \mathbb{P}^2$ where $F = F(x, y, z)$ is a (nonzero) degree 1 homogeneous polyonmial over k, and a *conic* in $\mathbb{P}^2$ is a subset of the form $V(G) \subset \mathbb{P}^2$ where $G = G(x, y, z)$ is a (nonzero) degree 2 homogeneous polynomial over k. Solve the following set of questions.
  (a) Show that there exists a bijection between the set of all lines passing through a fixed point in $\mathbb{P}^2$ and the set of points of $\mathbb{P}^1$.
  (b) Show that there exists a bijection between the set of all conics in $\mathbb{P}^2$ and the set of points of $\mathbb{P}^5$.
  (c) Prove that if a line and a conic has 3 (or more) points of $\mathbb{P}^2$ in common, then the line is contained in the conic.
  (d) Prove that there exists a conic passing through any 5 points of $\mathbb{P}^2$.
  (e) Prove that there exists an *irreducible* conic (that is, the defining equation is irreducible) passing through 5 points of $\mathbb{P}^2$, if and only if, no 3 of these points lie on a line in $\mathbb{P}^2$.
  (2+3+3+3+6=17 marks)